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Elasto-plastic analysis of an internally pressurized thick-walled cylinder using a strain gradient plasticity theory

X.-L. Gao *

Department of Mechanical Engineering—Engineering Mechanics, Michigan Technological University, 1400 Townsend Drive, Houghton, MI 49931-1295, USA

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Abstract

An analytical solution for the stress, strain and displacement fields in an internally pressurized thick-walled cylinder of an elastic strain-hardening plastic material in the plane strain state is presented. A strain gradient plasticity theory is used to describe the constitutive behavior of the material undergoing plastic deformations, whereas the generalized Hooke's law is invoked to represent the material response in the elastic region. The solution gives explicit expressions for the stress, strain and displacement components. The inner radius of the cylinder enters these expressions not only in non-dimensional forms but also with its own dimensional identity, unlike classical plasticity-based solutions. As a result, the current solution can capture the size (strengthening) effect at the micron scale. The classical plasticity-based solution of the same problem is shown to be a special case of the present solution. Numerical results for the maximum effective stress in the cylinder wall are also provided to illustrate applications of the newly derived solution.

Keywords: Thick-walled cylinder; Stress analysis; Elastic–plastic deformation; Strain gradient plasticity; Size effect

1. Introduction

In the context of classical plasticity, the internally pressurized thick-walled cylinder problem has been well studied. Solutions of this problem have been derived in the literature using different constitutive models. For example, for elastic perfectly plastic cylinders (tubes) under internal pressures exact solutions were provided in Hill (1950) and Nadai (1950). A closed-form solution for an internally pressurized thick-walled cylinder of an elastic strain-hardening plastic material in the plane strain state was derived in Gao and Wei (1991), and an analytical solution for a similar problem in the plane stress state was developed in Gao (1992). However, in these solutions (based on classical elasticity and plasticity theories) the size of the cylinder is involved only in a non-dimensional fashion (and neither the inner nor the outer radius of the

* Tel.: +1-906-487-1898; fax: +1-906-487-2822.
E-mail address: xgao@mtu.edu (X.-L. Gao).

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cylinder enters the solutions individually). This is true not only for the (larger) elastic domain, but also for the (localized) plastic domain near the inner surface.

Expanding cavity models (e.g., Johnson, 1970; Yoffe, 1982) are often employed to describe stress responses of elastic–plastic materials to progressive indentation loading in indentation tests because of their simplicity. For planar indentations the existing expanding cavity models (ECMs) are based on Hill's (1950) solution for the quasi-static expansion of a cylindrical tube of an elastic perfectly-plastic material under an internal pressure. These ECMs have been found to break down for materials having appreciable strain-hardening characteristics (e.g., Tabor, 1986; Lawn, 1998). Furthermore, indentation tests have revealed that hardness, which is determined from the size of impression caused by indentation loads, is size dependent when the characteristic length of the impression is on the order of tens of microns (e.g., Tabor, 1986; Hutchinson, 2000; Swadener et al., 2002). That is, the material hardness increases with the decrease of the size of the indent at the micron scale. However, the existing ECMs do not have capabilities to account for this indentation size effect, because Hill's (1950) solution upon which the ECMs were built was derived using classical plasticity that is local in nature and does not contain any internal length scale (Hutchinson, 2000). Therefore, higher-order (non-local) continuum theories should be applied to derive new solutions for the thick-walled cylinder problem so that improved ECMs, which incorporate both the strain-hardening and indentation size effects, can be ultimately constructed for modeling planar indentations.

The objective of the present paper is to provide such a solution using a strain gradient plasticity theory. In the derivation, the classical elasticity theory (Hooke's law) is used in the elastic region, whereas a strain gradient plasticity theory is employed in the plastic region. The solution is derived in a closed form for the stress, strain and displacement components. There is one parameter involved in the solution, whose numerical values can be determined from its defining equation (one boundary condition) using a simple iterative procedure. The inner radius of the cylinder is shown to explicitly appear in the solution with its own dimensional identity, unlike that in a classical plasticity solution.

The rest of this paper is organized as follows. Section 2 begins with a brief review of various strain gradient theories, resulting in the adoption of a gradient theory for the current analysis. The boundary-value problem (BVP) is then formulated, which leads to a closed-form elasto-plastic solution based on the chosen strain gradient plasticity theory. The classical plasticity-based solution of the cylinder problem is recovered in Section 3 as a special case of the gradient plasticity-based solution. Also, numerical results for the maximum effective stress in the cylinder wall are presented there to illustrate applications of the newly derived solution. The paper concludes with several remarks in the fourth and last section.

2. Formulation of the boundary-value problem

2.1. Review of strain gradient plasticity theories

Classical plasticity theories are based on Cauchy's stress principle, which assumes that the stress state at a material point in a continuum is influenced only by the stress states of the points in the immediate neighborhood of the material point. These plasticity theories do not consider long-range interactions among material points and are therefore local in nature. Lacking an internal length parameter, classical plasticity theories cannot describe the size effect observed in numerous experiments involving a small length scale, which include indentation tests. This has motivated the development of strain gradient plasticity theories (Hutchinson, 2000).

There exist two categories of strain gradient plasticity theories. In the first category, including those of Fleck and Hutchinson (1993, 2001), Gao et al. (1999), Huang et al. (2000) and Hwang et al. (2002), higher-order stresses (conjugated to higher-order strain gradients), in addition to the Cauchy stress, are introduced, and higher-order (or additional) governing equations and extra boundary conditions are necessitated. In the second category, including the ones proposed by Mühlhaus and Aifantis (1991), Acharya and Bassani (1996), Chen and Wang (2000) and Gao and Huang (2001), no higher-order stress is involved and the balance laws remain the same as those in classical plasticity. Moreover, the last three of the four strain gradient theories in the second category listed above are not accompanied by extra (higher-order) boundary conditions in their original formulations. This feature would make the three theories very attractive in applications, since higher-order boundary conditions may not be uniquely defined and/or can be difficult to satisfy. However, the necessity of having extra boundary conditions in applying a lower-order gradient theory in the second category has recently been demonstrated by Volokh and Hutchinson (2002). In contrast, the lower-order gradient theory of Mühlhaus and Aifantis (1991), also belonging to the second category and cited earlier, does involve extra boundary conditions, which are defined (non-uniquely) by a variational principle. This theory has recently been used in Gao (2002, 2003) to obtain analytical solutions of two different problems.

The strain gradient plasticity theory elucidated in Mühlhaus and Aifantis (1991) introduces higher-order spatial gradients of the effective plastic strain into the yield condition (or the constitutive equation for the flow stress), while leaving all other features of classical plasticity unaltered. The idea of modifying the standard yield criterion in the afore-mentioned manner was explored earlier in Coleman and Hodgdon (1985). This modification results in the inclusion of a length scale into classical plasticity and renders it possible to model mechanical phenomena involving fine length scales. The simplest version of this strain gradient plasticity theory in its rate-independent form (e.g., Zhu et al., 1997; Gao, 2002) utilizes, in the yield condition,

$$\sigma_e = \sigma_e^H - c\nabla^2 \varepsilon_e, \tag{1}$$

where σ_e and σ_e^H are, respectively, the total and the homogeneous part of the effective stress, ε_e is the effective plastic strain, c is the strain gradient coefficient (i.e., a force-like constant measuring the effect of strain gradient), and ∇^2 is the Laplacian operator. The modified yield criterion then reads $\sigma_e \leqslant \sigma_y$ (rather than $\sigma_e^H \leqslant \sigma_y$), where σ_y is the yield stress of the material; when the equality is reached, yielding will occur. The extra boundary conditions arising from the inclusion of the strain gradient term in Eq. (1) can be represented by

$$\frac{\partial \varepsilon_e}{\partial n} = 0 \quad \text{or} \quad \varepsilon_e = \overline{\varepsilon_e} \quad \text{on } \partial^P B, \tag{2}$$

where $\partial^P B$ is the boundary of the plastic region, n is the unit outward normal to $\partial^P B$, and the overbar stands for the prescribed value. All other equations in classical plasticity will remain unchanged in this strain gradient plasticity theory.

Eqs. (1) and (2) will be used, together with other relations in Hencky's deformation theory of plasticity (e.g., Mendelson, 1968; Gao, 1994, 1998, 1999), in deriving the solution for the plastic region.

2.2. Formulation

Consider an elasto-plastic plane strain cylinder of the inner radius a and outer radius b and subjected to the internal pressure p_i, as shown in Fig. 1. The usual polar coordinates (r, θ) will be used to represent a point on a cross section of the cylinder.

If p_i is sufficiently small, the entire cylinder remains elastic. However, when p_i becomes large enough, the cylinder begins to yield from the inner surface $r = a$. With the continuous increase of p_i the yielded region will expand outwardly. From symmetry it follows that the elasto-plastic interface is also a cylindrical surface for any value of p_i that produces a plastic region. Let r_c be the radius of this elasto-plastic interface, and p_c be the associated pressure acting on the interface under p_i (a generic value). Then, the material in the region $a \leqslant r \leqslant r_c$ is in the plastic state, whereas the material in the region $r_c \leqslant r \leqslant b$ remains elastic under p_i.

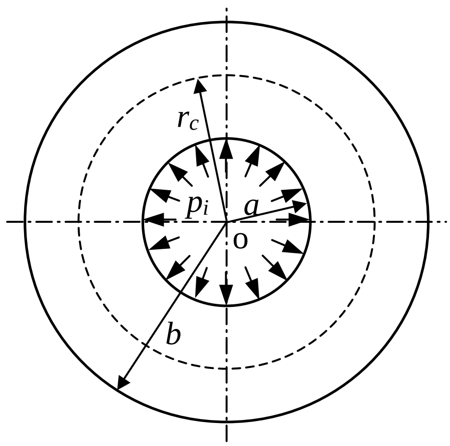

Fig. 1. Problem configuration.

When in a complex stress state, the power-law hardening model in classical plasticity has the form:

$$\sigma_e^H = \kappa \varepsilon_e^m, \tag{3}$$

where κ and m are two material constants. It can be shown that $\kappa = \sigma_y^{1-m} E^m$, with E being Young's modulus of the material. Eq. (3) will be used in Eq. (1) (as the expression for σ_e^H) to define the total effective stress σ_e in the strain gradient plasticity theory, which is to be applied to derive the solution in the plastic region. On the other hand, the material response in the elastic region is assumed to still obey the generalized Hooke's law. This will allow the direct application of Lamé's classical elasticity solution in the elastic region, which is away from the (localized) plastic zone near the inner surface.

The elasto-plastic cylinder problem can now be formulated as a standard BVP and solved analytically.

2.2.1. Solution in the elastic region ($r_c \leqslant r \leqslant b$)

This region can be treated as a thick-walled cylinder of the inner radius $r = r_c$ and outer radius $r = b$ and subjected to the internal pressure p_c. Then, using Lamé's solution for the plane strain case (e.g., Little, 1973) gives the stress components as

$$\sigma_{rr} = \frac{p_c r_c^2}{b^2 - r_c^2}\left(1 - \frac{b^2}{r^2}\right), \quad \sigma_{\theta\theta} = \frac{p_c r_c^2}{b^2 - r_c^2}\left(1 + \frac{b^2}{r^2}\right), \quad \sigma_{zz} = 2v\frac{p_c r_c^2}{b^2 - r_c^2}, \tag{4}$$

the strain components as

$$\begin{aligned} \varepsilon_{rr} &= \frac{1+v}{E}\frac{p_c r_c^2}{b^2 - r_c^2}\left(1 - 2v - \frac{b^2}{r^2}\right), \\ \varepsilon_{\theta\theta} &= \frac{1+v}{E}\frac{p_c r_c^2}{b^2 - r_c^2}\left(1 - 2v + \frac{b^2}{r^2}\right), \\ \varepsilon_{zz} &= 0, \end{aligned} \tag{5}$$

and the displacement component as

$$u = \frac{1+v}{E}\frac{p_c r_c^2}{b^2 - r_c^2}\left(1 - 2v + \frac{b^2}{r^2}\right) r. \tag{6}$$

In Eqs. (5) and (6), v is Poisson's ratio of the material. Clearly, this solution is expressed in terms of two yet-unknown parameters p_c and r_c, which will be determined from the boundary (interface) conditions later.

On the elasto-plastic interface $r = r_c$, the stress components (through the effective stress) given in Eq. (4) must satisfy the yield condition

$$\sigma_e|_{r=r_c} = \sigma_y. \tag{7}$$

This provides one relation for determining p_c and r_c.

2.2.2. Solution in the plastic region ($a \leqslant r \leqslant r_c$)

Under the assumptions of infinitesimal deformations, isotropic hardening, incompressibility and monotonic loading, the governing equations in a stress formulation, which embody Hencky's deformation theory, the strain gradient theory (with the modified von Mises' yield criterion), and the power-law hardening model, of the present axi-symmetric problem include the equilibrium equation (e.g., Chou and Pagano, 1967)

$$\sigma_{\theta\theta} - \sigma_{rr} = r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}, \tag{8}$$

the compatibility equation (e.g., Chou and Pagano, 1967)

$$r\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r} = \varepsilon_{rr} - \varepsilon_{\theta\theta}, \tag{9}$$

and the constitutive equations (e.g., Nadai, 1950; Gao, 1999, 2002)

$$\varepsilon_{rr} = \frac{3}{4}\frac{\varepsilon_e}{\sigma_e}(\sigma_{rr} - \sigma_{\theta\theta}) = -\varepsilon_{\theta\theta}, \tag{10}$$

$$\sigma_e = \kappa\varepsilon_e^m - c\nabla^2\varepsilon_e, \tag{11}$$

$$\sigma_e = \frac{\sqrt{3}}{2}(\sigma_{\theta\theta} - \sigma_{rr}). \tag{12}$$

The boundary conditions of this problem are

$$\sigma_{rr}|_{r=a} = -p_i, \quad \sigma_{rr}|_{r=r_c} = -p_c, \tag{13a, b}$$

$$\varepsilon_e|_{r=a} = D, \quad \varepsilon_e|_{r=r_c} = \frac{\sigma_y}{E}. \tag{14a, b}$$

Eq. (13a,b) are two standard boundary conditions for deriving a classical plasticity solution (e.g., Gao and Wei, 1991; Gao, 1992), while Eq. (14a,b) are two extra boundary conditions arising from the use of the strain gradient theory, as discussed earlier. Here, D is a constant to be determined as part of the solution.

Eqs. (8)–(12), (13a,b) and (14a,b) define the BVP for determining the stress and strain fields in the plastic region. This BVP can be analytically solved as follows.

Using Eq. (12) in Eq. (10) gives

$$\varepsilon_{rr} = -\frac{\sqrt{3}}{2}\varepsilon_e, \quad \varepsilon_{\theta\theta} = \frac{\sqrt{3}}{2}\varepsilon_e. \tag{15}$$

Substituting Eq. (15) into Eq. (9) then yields

$$\frac{\mathrm{d}\varepsilon_e}{\varepsilon_e} = -2\frac{\mathrm{d}r}{r}. \tag{16}$$

Integrating Eq. (16) from a to r and invoking Eq. (14a) will lead to

$$\varepsilon_e = D\frac{a^2}{r^2}. \tag{17}$$

Inserting Eq. (17) into Eq. (14b) gives

$$D = \frac{\sigma_y}{E}\frac{r_c^2}{a^2}. \tag{18}$$

Eq. (18) provides a first relation among the three unknown parameters D, r_c and p_c.

Using Eq. (17) in Eq. (11) results in

$$\sigma_e = \kappa D^m\left(\frac{a}{r}\right)^{2m} - 4cD\frac{a^2}{r^4}. \tag{19}$$

Now inserting Eqs. (12) and (19) into Eq. (8) gives

$$d\sigma_{rr} = \frac{2}{\sqrt{3}}\left(\kappa D^m\frac{a^{2m}}{r^{2m+1}} - 4cD\frac{a^2}{r^5}\right)dr. \tag{20}$$

A direct integration of Eq. (20) from a to r leads to, together with Eq. (13a),

$$\sigma_{rr} = -p_i + \frac{2}{\sqrt{3}}\left[\frac{\kappa D^m}{2m}\left(1 - \frac{a^{2m}}{r^{2m}}\right) - cDa^2\left(\frac{1}{a^4} - \frac{1}{r^4}\right)\right]. \tag{21}$$

Substituting Eq. (21) into Eq. (13b) results in

$$p_i - p_c = \frac{2}{\sqrt{3}}\left[\frac{\kappa D^m}{2m}\left(1 - \frac{a^{2m}}{r_c^{2m}}\right) - cDa^2\left(\frac{1}{a^4} - \frac{1}{r_c^4}\right)\right], \tag{22}$$

which is a second relation required for determining the three unknown parameters. The third relation is furnished by Eq. (7). Using Eqs. (4) and (12) in Eq. (7) yields

$$p_c = \frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{r_c^2}{b^2}\right) \tag{23}$$

as the last relation needed. Clearly, $0 \leqslant p_c < \sigma_y/\sqrt{3}$, as dictated by Eq. (23). By solving Eqs. (18), (22) and (23), D, r_c and p_c will be determined. Inserting Eqs. (18) and (23) into Eq. (22) gives

$$p_i = \frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{r_c^2}{b^2}\right) + \frac{2}{\sqrt{3}}\left[\frac{\kappa}{2m}\left(\frac{\sigma_y}{E}\right)^m\left(\frac{r_c^{2m}}{a^{2m}} - 1\right) - \frac{\sigma_y c}{E}\frac{1}{a^2}\frac{r_c^2}{a^2}\left(1 - \frac{a^4}{r_c^4}\right)\right]. \tag{24}$$

This equation may be solved by using an iterative procedure to obtain the value of r_c for given values of E, σ_y, κ, m, c (material properties) and a (inner radius). Knowing r_c, Eqs. (18) and (23) then give D and p_c, respectively. As displayed in Eqs. (18), (23) and (24), r_c, D and p_c are all dependent on the inner radius a. This will lead to the stress and strain fields which are varying with a, as shown below.

Using Eqs. (12), (18), (19), (21) and (24) gives the two in-plane stress components as

$$\begin{aligned}
\sigma_{rr} &= -\frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{r_c^2}{b^2}\right) + \frac{2}{\sqrt{3}}\left[\frac{\kappa}{2m}\left(\frac{\sigma_y}{E}\right)^m\left(1 - \frac{r_c^{2m}}{r^{2m}}\right) - \frac{\sigma_y c}{E}\frac{1}{a^2}\frac{r_c^2}{a^2}\left(\frac{a^4}{r_c^4} - \frac{a^4}{r^4}\right)\right], \\
\sigma_{\theta\theta} &= -\frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{r_c^2}{b^2}\right) + \frac{2}{\sqrt{3}}\left\{\frac{\kappa}{2m}\left(\frac{\sigma_y}{E}\right)^m\left[1 + (2m-1)\frac{r_c^{2m}}{r^{2m}}\right] - \frac{\sigma_y c}{E}\frac{1}{a^2}\frac{r_c^2}{a^2}\left(\frac{a^4}{r_c^4} + \frac{3a^4}{r^4}\right)\right\}.
\end{aligned} \tag{25}$$

It then follows from Eq. (25) that

$$\sigma_{zz} = \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}) = -\frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{r_c^2}{b^2}\right) + \frac{2}{\sqrt{3}}\left\{\frac{\kappa}{2m}\left(\frac{\sigma_y}{E}\right)^m\left[1 + (m-1)\frac{r_c^{2m}}{r^{2m}}\right] - \frac{\sigma_y c}{E}\frac{1}{a^2}\frac{r_c^2}{a^2}\left(\frac{a^4}{r_c^4} + \frac{a^4}{r^4}\right)\right\} \tag{26}$$

as the axial stress component. Substituting Eqs. (17) and (18) into Eq. (15) gives the strain components as

$$\varepsilon_{rr} = -\frac{\sqrt{3}}{2}\frac{\sigma_y}{E}\frac{r_c^2}{r^2}, \quad \varepsilon_{\theta\theta} = \frac{\sqrt{3}}{2}\frac{\sigma_y}{E}\frac{r_c^2}{r^2}, \quad \varepsilon_{zz} = 0. \tag{27}$$

Finally, from the geometric equations:

$$\varepsilon_{rr} = \frac{\mathrm{d}u}{\mathrm{d}r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} \tag{28}$$

and Eq. (27) it follows that the only non-zero (radial) displacement component is

$$u = \frac{\sqrt{3}}{2}\frac{\sigma_y}{E}\frac{r_c^2}{r}. \tag{29}$$

This completes the derivation of the solution in the plastic region ($a \leqslant r \leqslant r_c$).

With the parameters p_c and r_c determined respectively from Eqs. (23) and (24), the stress, strain and displacement components in the elastic region ($r_c \leqslant r \leqslant b$) can be readily obtained from Eqs. (4)–(6).

Clearly, the stress components given in Eqs. (25) and (26) explicitly depend on the inner radius a as well as the non-dimensional quantities r_c/a and a/r, noting that $r_c/r = (r_c/a)(a/r)$ and $r_c/b = (r_c/a)(a/b)$. This size dependence is also displayed by the strain and displacement components listed in Eqs. (27) and (29) as well as the stress, strain and displacement fields in the elastic region (see Eqs. (4)–(6)), because r_c involved in all relevant expressions is dependent on a, as shown in Eq. (24). Hence, the current solution has the capacity to account for the size effect. This is not the case for classical plasticity-based solutions. For example, in the elastic perfectly plastic solution of Nadai (1950) and in the elastic strain-hardening plastic solution of Gao and Wei (1991) for the same problem using the classical plasticity theory only non-dimensional geometrical quantities a/r and r_c/r are involved, and the inner radius a does not enter the solution individually, as will be seen in the next section.

3. Applications and numerical results

The elasto-plastic solution derived in the preceding section is applied here to obtain specific solutions and numerical results.

3.1. Classical plasticity solution

When $c = 0$, the BVP in the plastic region defined by Eqs. (8)–(12), (13a,b) and (14a,b) will be reduced to that based on the Hencky deformation theory and the von Mises yield criterion in classical plasticity. Hence, letting $c = 0$ in Eqs. (25) and (26) gives

$$\sigma_{rr} = \frac{\sigma_y}{\sqrt{3}}\left[-\left(1-\frac{r_c^2}{b^2}\right)+\frac{1}{m}\left(1-\frac{r_c^{2m}}{r^{2m}}\right)\right],$$
$$\sigma_{\theta\theta} = \frac{\sigma_y}{\sqrt{3}}\left[-\left(1-\frac{r_c^2}{b^2}\right)+\frac{1}{m}+\left(2-\frac{1}{m}\right)\frac{r_c^{2m}}{r^{2m}}\right], \tag{30}$$
$$\sigma_{zz} = \frac{\sigma_y}{\sqrt{3}}\left[-\left(1-\frac{r_c^2}{b^2}\right)+\frac{1}{m}+\left(1-\frac{1}{m}\right)\frac{r_c^{2m}}{r^{2m}}\right]$$

as the stress components, where r_c satisfies

$$p_i = \frac{\sigma_y}{\sqrt{3}}\left[\left(1-\frac{r_c^2}{b^2}\right)+\frac{1}{m}\left(\frac{r_c^{2m}}{a^{2m}}-1\right)\right], \tag{31}$$

which is reduced from Eq. (24) with $c = 0$. In deriving Eqs. (30) and (31), use has also been made of $\kappa = \sigma_y^{1-m}E^m$. The strain and displacement components in the plastic region have the same expressions as those listed in Eqs. (27) and (29) except that r_c involved in the expressions is now defined by Eq. (31). The same is true for the stress, strain and displacement components in the elastic region, which continue to be given by Eqs. (4)–(6) (with only r_c to be changed). This solution is identical with that derived in Gao and Wei (1991) for the same problem using classical plasticity. That is, the current gradient plasticity solution includes the classical plasticity-based solution as a special case. In particular, when $m = 0$, the elastic power-law plastic constitutive model employed here reduces to that for the elastic perfectly plastic material (see Eq. (3)). For this particular case it follows from Eqs. (30) and (31), with $m = 0$ and the use of l'Hôpital's rule, that

$$\sigma_{rr} = \frac{\sigma_y}{\sqrt{3}}\left[-1+\frac{r_c^2}{b^2}-2\ln\frac{r_c}{r}\right], \quad \sigma_{\theta\theta} = \frac{\sigma_y}{\sqrt{3}}\left[1+\frac{r_c^2}{b^2}-2\ln\frac{r_c}{r}\right], \quad \sigma_{zz} = \frac{\sigma_y}{\sqrt{3}}\left[\frac{r_c^2}{b^2}-2\ln\frac{r_c}{r}\right] \tag{32}$$

and

$$p_i = \frac{\sigma_y}{\sqrt{3}}\left[\left(1-\frac{r_c^2}{b^2}\right)+2\ln\frac{r_c}{a}\right]. \tag{33}$$

These expressions are identical to those given in Nadai (1950). Hence, the classical plasticity solution for the elastic perfectly plastic thick-walled cylinder is recovered by the current solution as a limiting case with $c = 0$ and $m = 0$.

For given loading (p_i), material properties (σ_y, m) and radius ratio (b/a), Eq. (31) can be readily solved to determine r_c/a. From Eqs. (30), (27), (29), (4)–(6) and (23) it can be seen that the stress, strain and displacement components of the resulting classical plasticity solution depend on the inner radius a only in a non-dimensional fashion (with $r_c/r = (r_c/a)(a/r)$ and $r_c/b = (r_c/a)(a/b)$), which differs from that displayed by the gradient solution with $c \neq 0$. Clearly, this property of the classical plasticity-based solution is also shown in the specific solution given in Eqs. (32) and (33).

3.2. Maximum effective stress

The effective stress (σ_e) measures the level of stress at a material point in a complex (multiaxial) stress state. It follows from Eqs. (18) and (19) that the maximum effective stress in the cylinder wall is given by, with $\sigma_e|_{r=a} = \sigma_e^{\max}$,

$$\sigma_e^{\max} = \sigma_y\left(\frac{r_c}{a}\right)^{2m} - 4\sigma_y\frac{c}{E}\frac{1}{a^2}\left(\frac{r_c}{a}\right)^2, \tag{34}$$

where use has been made of $\kappa = \sigma_y^{1-m}E^m$. Let

$$l \equiv \sqrt{\frac{c}{E}} \tag{35}$$

be a characteristic (internal) length, which is a material constant. Then, Eq. (34) can be rewritten as

$$\frac{\sigma_e^{max}}{\sigma_y} = \left(\frac{r_c}{a}\right)^{2m} - \frac{4l^2}{a^2}\left(\frac{r_c}{a}\right)^2. \tag{36}$$

The classical plasticity solution gives, using Eqs. (12) and (30),

$$\frac{\sigma_e^{max}}{\sigma_y} = \left(\frac{r_c}{a}\right)^{2m}, \tag{37}$$

which could also have been obtained directly from Eq. (34) with $c = 0$. A comparison of Eqs. (36) and (37) shows that the gradient plasticity solution incorporates an internal (material) length, l, and includes the inner radius, a, in both dimensional and non-dimensional forms, while the classical solution involves a only in a non-dimensional fashion. The differences between the two solutions in terms of the maximum effective stress are further illustrated in Fig. 2, which is based on the data listed in Table 1. In Fig. 2, Y denotes σ_e^{max}/σ_y. The material properties used to obtain the numerical values given in Table 1 are for an aluminum material having $E = 73$ GPa, $m = 0.25$, $c = 2.5$ N, and $l = 5.852(10^{-6})$ m (Zhu et al., 1997; Gao, 2002,

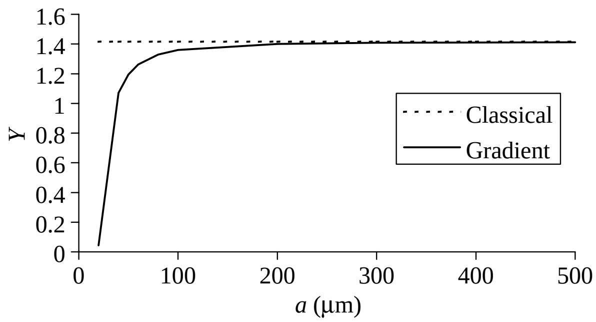

Fig. 2. Maximum effective stress in the cylinder wall.

Table 1
σ_e^{max}/σ_y for different values of a (with $r_c/a = 2.0$)

a (m)	σ_e^{max}/σ_y (gradient)	σ_e^{max}/σ_y (classical)
20×10^{-6}	0.04435054868	1.41421356237
40×10^{-6}	1.07174780895	1.41421356237
60×10^{-6}	1.26200656085	1.41421356237
80×10^{-6}	1.32859712402	1.41421356237
100×10^{-6}	1.35941904183	1.41421356237
200×10^{-6}	1.40051493224	1.41421356237
500×10^{-6}	1.41202178155	1.41421356237
0.001	1.41366561717	1.41421356237
0.002	1.41407657607	1.41421356237
0.005	1.41419164457	1.41421356237
0.01	1.41420808292	1.41421356237
0.05	1.41421334320	1.41421356237
0.1	1.41421350758	1.41421356237

2003). The listed data are directly calculated from Eqs. (36) and (37) using *Mathematica* program (of Wolfram Research, Inc.).

From Table 1 and Fig. 2, it is clear that the maximum effective stress predicted by the gradient plasticity solution is indeed size dependent when the inner radius a is very small (on the order of tens of microns). The smaller a is, the smaller σ_e^{max} becomes, thereby explaining the size (strengthening) effect at the micron scale. On the other hand, when the inner radius a is large (at the scale of 200 microns or above), the prediction of the gradient plasticity solution approaches that of the classical plasticity solution, which is a constant independent of a. This indicates that there is no pronounced size effect if no small (micron) length scale is involved, which is in agreement with what was noted by Hutchinson (2000) in a general context. Therefore, the use of classical plasticity to describe macroscopic behavior (beyond micron scale) of internally pressurized thick-walled cylinders is justified.

4. Conclusion

An analytical solution is provided for the plane strain problem of an internally pressurized thick-walled cylinder of an elastic strain-hardening plastic material. The solution is based on a strain gradient plasticity theory, which introduces a higher-order spatial gradient (the Laplacian) of the effective plastic strain into the standard expression of von Mises' yield condition in classical plasticity. Hencky's deformation theory, the modified von Mises yield criterion, and the power-law hardening model form the constitutive relations governing deformations in the plastic region. The generalized Hooke's law is used to describe material behavior in the elastic region.

The solution is derived in a closed form. Explicit expressions are provided for the stress, strain and displacement components in both the elastic and plastic regions. There is only one parameter involved in the solution that needs to be determined numerically using an iterative procedure. These expressions show that the cylinder inner radius enters the current solution with its own dimensional identity as well as in non-dimensional forms, unlike solutions based on classical elasticity and plasticity theories.

The classical plasticity-based solution of the same cylinder problem is recovered by the current gradient plasticity-based solution. Illustrative numerical results for the maximum effective stress in the cylinder wall are presented to show how the two solutions may differ. It is found that the gradient solution can capture the size effect, whereas the classical solution does not have the same capability. Nevertheless, the numerical data demonstrate that the classical plasticity-based solution and the gradient plasticity-based solution predict almost identical results if no small (micron) length scale is involved.

The analysis presented here is for pressurized cylinders undergoing plane strain deformations (e.g., long cylinders with very small longitudinal displacements). For cylinders in the plane stress state (e.g., open-ended gun barrels), the analysis based on the strain gradient plasticity theory is expected to be more complex, since the von Mises effective stress no longer has the linear form given in Eq. (12) (e.g., Gao, 1992). Whether a closed-form solution can be obtained for the plane stress counterpart problem using the same strain gradient theory remains to be investigated.

The newly derived solution can be used to construct improved ECMs, which incorporate both the strain-hardening and indentation size effects, for characterizing planar indentations. This is currently being pursued.

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